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CONNECTED $[a,b]$-FACTORS IN GRAPHS

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In this paper, we prove the following result:

Let G be a connected graph of order n, and minimum degree δ. Let a and b two integers such that $2a \le b$. Suppose $n \ge \frac{(a+b)(a+b-1)}{b}$ and $\delta \ge \frac{n}{1+\left\lfloor \frac{b}{a} \right\rfloor}$.

Then G has a connected $[a,b]$-factor.

I. Introduction

We consider simple graphs without loops. Let G be a graph of order n, with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(x)$ the degree of any vertex x in G, by $\delta(G)$ (resp. $\Delta(G)$) the minimum (resp. maximum) degree of G and if G is not complete then $\sigma_2(G) = \min\{d_G(u)+d_G(v) \mid u,v \in V(G),\ u,v$ nonadjacent $\}$.

Let $a \le b$ be positive integers. Recall that $\left\lfloor \frac{b}{a} \right\rfloor$ (resp. $\left\lceil \frac{b}{a} \right\rceil$) is the greatest (resp. the smallest) integer such that $\left\lfloor \frac{b}{a} \right\rfloor \le \frac{b}{a}$ (resp. $\frac{b}{a} \le \left\lceil \frac{b}{a} \right\rceil$).

A spanning subgraph F of G is called an $[a,b]$-factor of G if $a \le d_F(x) \le b$ holds for all $x \in V(G)$. If F is connected then the factor is said to be connected. If H is a $[a-1,b]$ factor of G, we say that a vertex x is saturated if $d_H(x) \ge a$.

Many authors have worked on $[a,b]$-factors as it can be seen in the bibliography of [1]. When the connectedness is not required, there exists a sufficient

Mathematics Subject Classification (2000): 05C70 and necessary condition for the existence of an $[a,b]$-factor in a graph G [6]: "G has an $[a,b]$-factor if and only if $b|S| - a|T| + \sum_{v\in T} d_{G\setminus S}(v) \geq 0$, for all disjoint subsets S,T of $V(G)$."

The concept of connected factors was introduced by M. Kano [4]. This topic is close to the hamiltonian problem as a connected 2-factor is obviously an hamiltonian cycle. On the other hand, we remark that a connected $k-$factor is a connected $k-$regular spanning subgraph.

In [3], M. Kano gives the following conjecture:

"Let $2 \leq a < b$ be integers and G be a connected graph with $\delta(G) \geq a$ and $n \geq a+b+1$. If $\sigma_2(G) \geq \frac{2an}{a+b}$, then G has a connected $[a,b]$-factor."

Counterexamples to this conjecture have been given by Y. Li and M. Cai [5], proving that the bound $a+b+1$ is too low.

In this paper, we prove the conjecture under the stronger condition: $\delta(G) \geq \frac{n}{1+\left[\frac{b}{a}\right]}$.

II. Existence of a connected $[a,b]$-factor in a graph

Case $a=2$

M. Kano announced that he proved the following result [3], however the proof has never been published.

Theorem 2.1. *Let $b \geq 3$ be an integer and let G be a connected graph of order n and minimum degree $\delta \geq 2$.*
If $\sigma_2(G) \geq \frac{4n}{b+2}$, then G has a connected $[2,b]$-factor.

This theorem is a corollary of one of our results [7] that we give below.

Theorem A. *Let G be a 2 edge-connected graph, $b \geq 2$ be an integer.*
Suppose $\sigma_2(G) \geq \frac{4n}{b+2}$, then G has a 2 edge connected $[2,b]-$factor.

Proof of the Theorem 2.1. The proof is by induction on the number r of cutedges of G.

If $r=0$, thus G is 2 edge-connected and we are done by Theorem A.

Suppose now $r \geq 1$. Let xy be a cut-edge between two components C_1 and C_2 and as $\delta \geq 2$, no one of these components is reduced to a single vertex. We contract the edge xy into a vertex z. We get a graph G_1 with $(r-1)$

cut-edges. no degree has decreased, so the induction hypothesis is fulfilled, and G_1 has a $[2,b]$-connected factor F_1.

The edges of F_1, joined to the edge xy induce a connected factor F in G.

We have $2 \le d_{F_1}(z) \le b$. Then, by the connectivity of F_1, we get $d_F(x) \ge 2$, $d_F(y) \ge 2$, and $d_F(x) + d_F(y) - 2 \le b$ so $d_F(x) + d_F(y) \le b+2$. Thus F is a $[2,b]$-connected factor. ∎

Case $b/a \in \mathbf{N}$

Theorem 2.2. *Let a and b be two integers such that $2 \le a < b$, a divides b. Let G be a connected graph of order n, and minimum degree δ.*

If (i) $n \ge \frac{(a+b)(a+b-1)}{b}$ *and* (ii) $\delta \ge \frac{an}{a+b}$,

then G has a connected $[a,b]$-factor.

Let us give first two remarks.

Remark 1. The bound on δ is best possible. Let a and b be two integers, such that a divides b, $p \ge 1$ and $q = \frac{b}{a}p + 1$. Then $p = \frac{a(q-1)}{b}$. Consider the graph $K_{p,q}$ and note that its minimum degree is equal to $p < \frac{aq}{b} < \frac{an}{a+b}$. The difference between $\frac{an}{a+b}$ and p is tight: $\frac{an}{a+b} - p = \frac{a}{a+b} < 1$. The condition (i) is easily fulfilled. On the other hand, if e is the number of edges of a connected $[a,b]$-factor in the graph, it is clear that $qa \le e \le pb$ holds, which contradicts the choice of p and q. So, $K_{p,q}$ has no $[a,b]-$factor.

Remark 2. We now show that the bound on n in the theorem is best possible when the bound on δ is fixed.

Suppose that $a+b$ divides n and $\frac{bn}{a+b}$ odd. Let G be a graph the vertex set of which consists of two parts, A of cardinality $\frac{an}{a+b} - 1$ and B of cardinality $\frac{bn}{a+b} + 1$; we join A and B by a complete bipartite graph and we add a perfect matching in B. Thus G has n vertices and $\delta = \frac{an}{a+b}$.

Suppose G contains an $[a,b]$-factor F. In F, each vertex of B sends at least $a-1$ edges towards A. On the other hand, there exist at most $b|A|$ edges in F going out of A. Thus we obtain:

$(a-1)|B| \le b|A|$, so $(a-1)\left(\frac{bn}{a+b}\right) \le b\left(\frac{an}{a+b} - 1\right)$, which gives the condition $n \ge \frac{(a+b)(a+b-1)}{b}$.

Proof of the theorem. For $a = 2$, the result is true by Theorem 2.1. From now on, let $a \ge 3$.

Note that $b \geq 2a \geq a+3$, since $b/a \in \mathbf{N}$. We prove the theorem by induction on a.

Suppose that the result is true for the integer $a-1$ and for any $b \geq 2a-2$. From (ii), $\delta(G) \geq \left\lceil \frac{an}{a+b} \right\rceil = \left\lceil \frac{(a-1)n}{(a-1)+(b-b/a)} \right\rceil$, where $a-1$ divides $b-b/a$.

By the induction hypothesis, we obtain that G has a connected $[a-1, b-b/a]$-factor, say F_0. For each $[a-1,b]$-factor F, we set $X_F = \{x | \ d_F(x) = a-1\}$. We add edges to F_0 in order to obtain a larger connected $[a-1,b]$-factor, say F_1, with the following constraints:

(1) X_{F_1} is of minimum cardinality among all the X_F, where F is any connected $[a-1,b]$-factor F such that $E(F_0) \subset E(F)$;

(2) if the assertion (1) is satisfied, then $|E(F_1)| - |E(F_0)|$ is minimum.

Let X (resp. Y, Z) denote the set of vertices whose degree in F_1 is $a-1$ (resp. b, between a and $b-1$). Note that, by the minimality of $|X|$, there is no edge of $G \setminus F_1$ between X and Z.

If $X = \emptyset$, then the result is proved. Otherwise, we choose a vertex x_1 in X.

Inequalities (i) and (ii) imply

$$(*) \qquad \delta \geq \left\lceil \frac{a(a+b-1)}{b} \right\rceil = a + \left\lceil \frac{a(a-1)}{b} \right\rceil \geq a+1$$

thus x_1 has at least two incident edges not in F_1 and we can suppose that the corresponding neighbors have degree b in F_1 otherwise we can increase the degree of x_1 by one.

Let Y_1 be the subset of Y consisting of the neighbors of x_1 in $G \setminus F_1$, and $Y_2 = Y \setminus Y_1$. Denote by Z_1 the set of vertices in Z of degree a in F_1 and $Z_2 = Z \setminus Z_1$.

To prove now the result, we need some claims.

For any factor F and $u \in V(G)$, denote by $\Delta_F(u)$ the integer $d_F(u) - (b - b/a)$ and $\Delta_F = \sum_{\Delta_F(u)>0} \Delta_F(u)$. Note that ($u \in Z_1 \cup X$ or $d_F(u) = a+1$) implies $\Delta_F(u) \leq 0$.

Claim 1. $|Z| \geq |Z_1| \geq \frac{b}{a}|Y|$.

Proof. From F_0 to F_1, the degrees of the vertices in Y increase by exactly b/a. By (2):

—there is no edge of $F_1 \setminus F_0$ inside $Y \cup Z_2$.

—for each $u \in Y \cup Z_2$, at least $\Delta_{F_1}(u)$ edges in $F_1 \setminus F_0$, incident to u have their other extremity in Z_1.

As the minimum degree in F_0 is $(a-1)$, each vertex of Z_1 is incident to at most one edge of $F_1 \setminus F_0$. So, in particular, we have

$$(**) \qquad |Z_1| \geq \sum_{y\in Y} \Delta_{F_1}(y) \ \geq \frac{b}{a}|Y|. \qquad \blacksquare$$

Let us define property $\mathcal{P}$:

For any connected $[a-1,b]$-factor F, we can define a partition of $V(G)$ into X,Y,Z as in F_1.

We say that F has **property $\boldsymbol{\mathcal{P}}$** if F satisfies the following two assertions:

$\boldsymbol{\alpha}$) For each vertex $u\in V(G)$, we can define a set $Z(u)\subset Z_1\cap N_F(u)$ such that these sets satisfy: $Z_F=\sum_{u\in V(G)}|Z(u)|\geq \Delta_F$;

$\boldsymbol{\beta}$) If $u\neq u'$ then $Z(u)\cap Z(u')=\emptyset$.

For the factor F_1, set $Z(u)=N_{F_1\setminus F_0}(u)$ if $\Delta_{F_1}(u)>0$ and $Z(u)=\emptyset$ if $\Delta_{F_1}(u)\leq 0$. Then F_1 satisfies property $\mathcal{P}$ by the proof of Claim 1.

We consider now a connected factor F satisfying $\mathcal{P}$ and furthermore, among all the factors satisfying $\mathcal{P}$, F is chosen such that:

(1) $|X|$ is minimum;

(2) the assertion (1) being satisfied, $|E(F)|$ is minimum.

Note that this factor exists, as F_1 has property $\mathcal{P}$.

It follows from property $\mathcal{P}$ for F that the inequalities $(**)$ are still true and Claim 1 is again satisfied; thus we have:

$$(***) \qquad |Y| < \frac{an}{a+b}, \quad \text{otherwise} \quad |Y|+|Z|\geq n \quad \text{and} \quad X=\emptyset.$$

Remark. In all further constructions (Claims 2 to 7), the sets X,Y,Z are modified, and we **need to verify property** $\mathcal{P}$ at each time so that Claim 1 is satisfied and thus **inequality** $(***)$ **is still true.**

Let $x\in X$ and $Y_1=N_{G\setminus F}(x)$. Note that $Y_1\subset Y$, otherwise, if xu is an edge not in F, with $u\in X\cup Z$, we saturate x, we put x in $Z(u)$ and $\mathcal{P}$ is still satisfied.

We denote by δ_0 the bound $\left\lceil \frac{an}{a+b}\right\rceil$, and set $|Y|=\delta_0-s$ and $|Y_1|=\delta_0-s_1$. Thus $1\leq s\leq s_1$. As $\delta_0\leq d_G(x)=d_F(x)+d_{G\setminus F}(x)=a-1+|Y_1|$, we get $s_1\leq a-1$. Furthermore, we recall that $|Y_1|\geq 2$ as $\delta\geq a+1$ (inequality $(*)$).

Claim 2. *Y_1 is a stable set in F, $Y\cup Z_2$ is a forest in the factor F and there is no cycle in F with an edge inside Y_2.*

Proof. If there exists an edge in F between 2 vertices of Y_1, say u and v, we delete this edge and add xu and xv, thus we get another connected factor satisfying property $\mathcal{P}$ with smaller $|X|$, a contradiction with the minimality of $|X|$.

If we have a cycle with an edge internal to $Y\cup Z_2$, we suppress this edge then we have a contradiction with the minimality of $|E(F)|$. ∎

Denote by $\epsilon_F(U,V)$ the number of edges in F adjacent to both two disjoint subsets U and V. Let $\mathcal{F}_Y$ be the remaining of the forest in F, constructed on the vertices of Y after deletion of the trivial components contained in Y_1. Let $|\mathcal{F}_Y|$ be its size, p be the number of its components and $p_1=|V(\mathcal{F}_Y)\cap Y_1|$.

Claim 3.

1) *Any component γ of the forest $\mathcal{F}_Y$ has at most one vertex in Y_1 and $|\mathcal{F}_Y|=|Y_2|-p+p_1$.*

2) *Let $\{y_1\}=\gamma\cap Y_1$ (when y_1 exists), then any path $P_F(x,\gamma)$ in F reaches γ in y_1.*

3) $\epsilon_F(x,Y)+|\mathcal{F}_Y|\leq|Y_2|$.

Proof. 1) If $V(\gamma)\cap Y_1$ contains $\{y_1,y_2\}$, we may suppose that a path $P_F(x,\gamma)$ reaches γ in $u\neq y_1$; then we add xy_1 to the factor F and we delete the edge y_1t contained in the path $P_F[y_1,u]$ in the component γ. We add x to $Z(y_1)$, then $Z(t),\Delta(t)$ and $Z(x)$ do not change. The new connected factor has again the property $\mathcal{P}$. We have a contradiction with the minimality of $|X|$.

The equality is immediate.

2) If the assertion 2) is not true, then let u be the vertex where $P_F(x,\gamma)$ reaches γ. As in the first part, in the component γ, we delete the edge incident to y_1 in the path joining y_1 to u; then we add xy_1. We get a contradiction with the minimality of $|X|$.

3) The assertion 3) is equivalent to $\epsilon_F(x,Y)\leq p-p_1$. In the factor F the vertex x is not adjacent to two vertices of the same component γ of $\mathcal{F}_Y$ (by the condition (2) of the definition of F). We show, if $\gamma\cap Y_1\neq\emptyset$, that the vertex x is not adjacent to γ if $\gamma\cap Y_1\neq\emptyset$. Let us set $\{y_1\}=\gamma\cap Y_1$; we know that $xy_1\notin E(F)$. In fact, if x is adjacent to a vertex $v\in\gamma$, we add the edge xy_1, and remove the edge y_1u, which belongs to the path $P_F[y_1,v]$ in γ. The property $\mathcal{P}$ is still satisfied and we have a contradiction with the minimality of $|X|$. ∎

Claim 4. *Let $z\in Z_2$. Then $\epsilon_F(z,Y_1)\leq 1$.*

Proof. Suppose that $\epsilon_F(z,Y_1)\geq 2$. Let u and v be the two neighbors of z in F belonging to Y_1. Then, in F, there exists a path between x and z, avoiding for example zu. We remove zu, and add xu to F. Since only $\Delta(z)$ decreases, $Z(u)$, $\Delta(u)$ and $Z(z)$ remain the same. We have a contradiction with the minimality of $|X|$. ∎

Claim 5. *In the factor F, let $\mathcal{C}_1$ be the subgraph generated by the components of $\mathcal{F}_Y$ intersecting Y_1.*

1) Let $z \in Z_1$. Then $\epsilon_F(z, \mathcal{C}_1 \cap Y_2) \leq 1$.

2) Let $z \in Z_2$. Then $\epsilon_F(z, \mathcal{C}_1 \cup Y_1) \leq 1$ and $|\mathcal{F}_Y| + \epsilon_F(z, Y) \leq |Y_2| + 1$.

Proof. Note that any vertex z of Z is not adjacent to two different vertices of the same component of the forest (by Claim 3).

∘ Suppose that $z \in Z$ is adjacent to u in a component γ, and to u' in a second component γ' of $\mathcal{C}_1$ with $u, u' \in Y_2$. In F, consider a path from x to $A = \gamma \cup \gamma' \cup \{z\}$ and we may assume, w.l.o.g. that this path reaches A in z or γ'. Let $\{y_1\} = \gamma \cap Y_1$ and $\{y_1'\} = \gamma' \cap Y_1$. Then we delete the edge $y_1 t$ in $P_\gamma(y_1, u)$ from the factor F and we add the edge xy_1. The factor we get is still connected and satisfies $\mathcal{P}$ as $|Z(y_1)| + |Z(t)|$ increases and $\Delta(y_1) + \Delta(t)$ decreases and we have a contradiction with the minimality of $|X|$.

Thus we proved that for any $z \in Z$, $\epsilon_F(z, \mathcal{C}_1 \cap Y_2) \leq 1$. In particular, the inequality is satisfied by any $z \in Z_1$.

∘ Suppose there exists $z \in Z_2$ such that $\epsilon_F(z, \mathcal{C}_1 \cup Y_1) \geq 2$. By part 1), two neighbors of z are not both in Y_2. By Claim 4, $\epsilon_F(z, Y_1) \leq 1$. So, in F, z has at least a neighbor u' in $\mathcal{C}_1 \cap Y_2$ and another one, u, in Y_1. Thus $u' \in \gamma' \cap Y_2$ where γ' is a component of $\mathcal{C}_1$.

In both cases, $u \in \mathcal{C}_1 \cap Y_1$ or otherwise, we set $A = \gamma' \cup \{u\} \cup \{z\}$ and $y_1' = \gamma' \cap Y_1$.

If a path $P_F(x, A)$ reaches A in z or u, we delete the edge $y_1' t$ in the path $P_F[y_1', u']$ contained in γ' and we add xy_1'.

If $P_F(x, A)$ reaches A in y_1', we delete the edge zu and we add xu.

In any case we saturate x and we verify that Δ_F remains the same or decreases by one and Z_F does not decrease. So, the new connected factor satisfies $\mathcal{P}$, and we have a contradiction with the minimality of $|X|$.

Then, we immediately get, for $z \in Z_2$

$$|\mathcal{F}_Y| + \epsilon_F(z, Y) \leq (|Y_2| - p + p_1) + (p - p_1 + 1) = |Y_2| + 1.$$ ∎

Claim 6.

1) *For any $x' \in X$,* (1) $\epsilon_F(x', Y) \leq (a - s - 1)$.

2) $\epsilon_{G \setminus F}(z, G \setminus Y) = 0$ *for any $z \in Z_1$ such that $\epsilon_F(z, Y_1) \geq 2$.*

(2) $\epsilon_F(z_1,Y)\leq a-s$ *for any* $z_1\in Z_1$.

Proof. 1) For each $x'\in X$, $d_F(x')=a-1$, $|Y|=\delta_0-s$. All edges $x't$ of G with $t\notin Y$ are edges of F; otherwise we add an edge $x't$, with $t\notin Y$, we put x' in $Z(t)$ and then obtain a new factor F'.

∘ if $t\in Z_1$ and $t\in Z(u)$ for some u, $\Delta_{F'}=\Delta_F$ and $|Z(u)|+|Z(t)|$ invariant;

∘ if $t\notin Z_1$, Δ_F increases by at most 1, and $|Z(t)|$ increases by 1. Thus the new connected factor F' satisfies property $\mathcal{P}$ and we get a contradiction with the minimality of $|X|$. So, $\delta_0\leq d(x')=d_Y(x')+d_{G\setminus Y}(x')\leq\delta_0-s+d_{G\setminus Y}(x')$ thus

$$(1)\qquad d_{G\setminus Y}(x')\geq s \quad\text{and}\quad \epsilon_F(x',Y)\leq a-1-s.$$

2)-a) Let u_1 and u_2 be two neighbors of z in Y_1, in the factor F. Suppose zz', with $z'\notin Y$, is an edge of $G\setminus F$. A path $P_F(x,z)$ does not use for example zu_1. We delete zu_1 from the factor F, add xu_1 and zz' and put x into $Z(u_1)$, then we obtain a new connected factor F'. We have two cases:

∘ If z' was in Z_2, $\Delta(z')$ increases by one, Δ_F by at most 1, and $|Z(u_1)|$ increases by 1.

∘ If z' was in Z_1, it is now in Z_2 with $\Delta_{F'}(z')\leq 0$ and then $\Delta_{F'}=\Delta_F$. If z was in $Z(u_1)$, we put it in $Z(u_2)$, and $Z_{F'}=Z_F$.

If z' was in $Z(v)$ for some v, Z_F remains the same or increases by 1.

In any case, $\mathcal{P}$ is still satisfied by F' and we obtain a contradiction with the minimality of $|X|$.

-b) Let $z_1\in Z_1$. Suppose $\epsilon_F(z_1,Y)\geq a-(s-1)\geq s_1-s+2=|Y_2|+2$. Then z_1 has at least two neighbors in Y_1 in the factor F. As $|Y|=\delta_0-s$, we have $\epsilon_G(z_1,G\setminus Y)\geq s$. As $d_F(z_1)=a$, we have $\epsilon_F(z_1,G\setminus Y)\leq s-1$. Thus there is at least one edge of $G\setminus F$, say zz' with $z'\notin Y$, a contradiction with the first part of 2). So we have inequality (2). ∎

Claim 7.

1) Let $z\in Z_1$ *such that* $\epsilon_F(z,Y_1)\geq 2$. *Then* $\epsilon_G(z,\mathcal{F}_Y)\leq p$.

2) If $z\in Z_1$ *and* $\epsilon_F(z,Y_1)\neq 1$, *then* $|\mathcal{F}_Y|+\epsilon_F(z,Y)\leq|Y_2|+a-s_1$.

Proof. 1) The proof is by contradiction so z has at least two neighbors in the same component γ of $\mathcal{F}_Y$. Let v be the vertex of the component γ where a path $P_F(z,\gamma)$ reaches γ. If z is adjacent in G to a vertex $u\in\gamma$, $u\neq v$, then zu is in $G\setminus F$ by Claim 2. Let y_1 and y_1' be two vertices of Y_1 which are neighbors of z in the factor F.

Case 1. $z\notin P_F(x,\gamma)$.

We may suppose that $P_F(z,\gamma)$ does not use zy_1. We delete from the factor F the edge zy_1, and we add xy_1. On the other hand, we add zu and delete the edge uu' of the path joining u and v in the component γ. The factor F remains connected. We put x in $Z(y_1)$ and we can suppose that z is now in $Z(y_1')$.

Case 2. $z \in P_F(x,\gamma)$.

We may suppose that zy_1 is not in $P_F(x,\gamma)$ or that this edge is not in the segment $[x,z]$ of the path. We do the same constructions as in the previous case.

In any case, it is easy to verify that Z_F increases by one and that Δ_F decreases by one. We have a contradiction with the minimality of $|X|$.

2) If $\epsilon_F(z,Y_1)=0$, the inequality is trivial.

Suppose now $\epsilon_F(z,Y_1)\geq 2$.

We have $\epsilon_F(z,Y)=\epsilon_F(z,Y\backslash\mathcal{F}_Y)+\epsilon_F(z,\mathcal{F}_Y)$ and $|V(Y\backslash\mathcal{F}_Y)|=\delta_0-(s_1+p_1)$ thus, in G, z has at least (s_1+p_1) neighbors out of the set $Y'=Y\setminus\mathcal{F}_Y$. So

$$\epsilon_F(z,G\setminus Y')\geq(s_1+p_1)-\epsilon_{G\backslash F}(z,G\setminus Y')$$

and then,

$$\epsilon_F(z,Y')=a-\epsilon_F(z,G\setminus Y')\leq a-(s_1+p_1)+\epsilon_{G\backslash F}(z,G\setminus Y').$$

We obtain:

$$\epsilon_F(z,Y)=\epsilon_F(z,Y')+\epsilon_F(z,\mathcal{F}_Y)\leq a-(s_1+p_1)+\epsilon_{G\backslash F}(z,G\backslash Y')+\epsilon_F(z,\mathcal{F}_Y).$$

But $\epsilon_{G\backslash F}(z,G\backslash Y')=\epsilon_{G\backslash F}(z,\mathcal{F}_Y)$ as $\epsilon_{G\backslash F}(z,G\backslash Y)=0$ by Claim 6 2).

We get immediately, using part 1) of the Claim:

$$\epsilon_F(z,Y)\leq(a-(s_1+p_1))+\epsilon_G(z,\mathcal{F}_Y)\leq(a-s_1)+(p-p_1).$$

So

$$\epsilon_F(z,Y)+|\mathcal{F}_Y|\leq|Y_2|+a-s_1.$$ ∎

End of the proof of the theorem.

We have the equalities:

$$b|Y|=\sum_{y\in Y}d_F(y)=\epsilon_F(Y,X)+\epsilon_F(Y,Z)+2|\mathcal{F}_Y|.$$

By Claim 6, we have $\epsilon_F(Y,X) \leq (a-s-1)|X|$ and $\epsilon_F(Z_1,Y) \leq (a-s)|Z_1|$ and by Claim 3,

$$\epsilon_F(x,Y) + |\mathcal{F}_Y| \leq |Y_2|. \tag{3}$$

∘ Suppose first that $|Z_2| \geq 1$. Let z be some vertex in Z_2.

$$\epsilon_F(z,Y) + |\mathcal{F}_Y| \leq |Y_2| + 1 \leq |Y_2| + (a - s_1). \tag{4}$$

Thus,

$$\begin{aligned} b|Y| \leq (a-s-1)(|X|-1) + \epsilon_F(x,Y) + (s_1 - s + 1)(|Z_2| - 1) \\ + \epsilon_F(z,Y) + (a-s)|Z_1| + 2|\mathcal{F}_Y|. \end{aligned}$$

Using inequalities (3) and (4), we get:

$$\begin{aligned} b|Y| \leq (a-s)(|X| + |Z_1|) - |X| - (a-s-1) \\ +(s_1 - s + 1)|Z_2| - (s_1 - s + 1) + 2|Y_2| + 1. \end{aligned}$$

Thus $b|Y| \leq (a-s)(n-|Y|-|Z_2|) - |X| + (s_1-s+1)|Z_2| + (s_1-a+1)$, which leads to:

$$(b+a-s)(\delta_0 - s) \leq (a-s)n - (1+|Z_2|)(a-s_1-1) - |X|$$

i.e.

$$\frac{bsn}{a+b} \leq s(a+b-s) - (1+|Z_2|)(a-s_1-1) - |X|$$

and then

$$n \leq \frac{(a+b-s)(a+b)}{b}\left(1 - \frac{(1+|Z_2|)(a-s_1-1) + |X|}{s(a+b-s)}\right).$$

If $|X| \neq \emptyset$, we obtain $n < \frac{(a+b-1)(a+b)}{b}$, a contradiction with the hypothesis (i) of the theorem.

∘ Suppose now that $|Z_2| = 0$.

By Claim 1, Z_1 is not empty. If there exists a vertex z in Z_1 such that $\epsilon_F(z,Y_1) \neq 1$, we apply 2) of Claim 7 and we have the same calculations as in the previous case.

Otherwise, $\epsilon_F(z,Y_1) = 1$ holds for each $z \in Z_1$, and $\epsilon_F(z,Y) \leq (p-p_1)+2$ by part 2) of Claim 5.

Thus

$$\epsilon_F(z,Y) + |\mathcal{F}_Y| \leq |Y_2| + 2 \tag{5}$$

holds.

Then we obtain:

$$b|Y| \le (a-s)(|X|+|Z_1|)-(|X|+1)-2(a-s)+\epsilon_F(z,Y)+\epsilon_F(x,Y)+2|\mathcal{F}_Y|.$$

Applying inequalities (3) and (5), we get

$$b|Y| \le (a-s)(n-|Y|)-(|X|-1)-2(a-s)+2(|Y_2|+1).$$

Thus $n \le \frac{(a+b-s)(a+b)}{b}\left(1-\frac{2(a-s)+(|X|-1)-2(s_1-s+1)}{s(a+b-s)}\right)$ which can be written as

$$n \le \frac{(a+b-s)(a+b)}{b}\left(1-\frac{2a-2s_1+(|X|-1)-2}{s(a+b-s)}\right),$$

and the conclusion is the same as previously, except eventually in the case when $|X|=1$, $s_1=a-1$, $s=1$.

We study this last case:

For any $z \in Z_1$, $\epsilon_F(z,Y_1)=1$, $|Y|=\delta_0-1$ and $|Z_1|=n-\delta_0$. So:

$$\begin{gathered}\epsilon_F(Y_1,Z_1)=|Z_1|=n-\delta_0 \le n-\frac{an}{a+b}=\frac{bn}{a+b}\\ \epsilon_F(Y_1,Y_2) \le |Y_2|-p+p_1=a-2-p+p_1 \le a-2\\ \epsilon_F(x,Y_1)=0.\end{gathered}$$

Thus $\epsilon_F(Y_1,G\setminus Y_1)=b|Y_1| \le \frac{bn}{a+b}+a-2$ which gives

$$b\left(\frac{an}{a+b}-1\right) \le b(\delta_0-1) \le \frac{bn}{a+b}+a-2$$

i.e.

$$n\left(\frac{ab}{a+b}-\frac{b}{a+b}\right) \le a+b-2,$$

and finally

$$n \le \frac{(a+b-2)(a+b)}{b(a-1)},$$

a contradiction with the hypothesis (i) on n. ∎

Case $b/a \notin \mathbf{N}$

Theorem 2.3. *Let a and b two integers such that $4 \leq 2a < b$.*
Let G be a connected graph of order n, and minimum degree δ.
If (i) $n \geq \frac{(a+b)(a+b-1)}{b}$ and (ii) $\delta \geq \frac{n}{1+\lfloor \frac{b}{a} \rfloor}$,

then G has a connected $[a,b]$-factor.

Proof. Let $b_1 = a\left\lfloor \frac{b}{a} \right\rfloor$, then $2a \leq b_1 < b$ and a divides b_1. On the other hand, $\delta \geq \frac{an}{a+b_1}$ and $n \geq \frac{(a+b)(a+b-1)}{b} \geq \frac{(a+b_1)(a+b_1-1)}{b_1}$.

The previous theorem allows us to say that G has a connected $[a,b_1]$-factor that is also a connected $[a,b]$-factor. ∎

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